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Creativity and the sensorimotor grounding of mathematics

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1 Introduction

Hearing the word "mathematics" often conjures up images of boring high school lectures and arduous problem sets. To many of us, mathematics seems like an overly abstract and seemingly untouchable subject matter, removed from any connection to the real world, and at the same time rigid and unwieldy. The mathematician Zagier (2012: 16) writes that "most people cannot begin to imagine how mathematics and pleasure can be related at all", and he says that some people are even "filled with dread at the mere thought of mathematics" (ibid. 12).

In this paper, we highlight the many ways in which math and creativity are connected. We argue that the inability to see the more creative aspects is part of what makes math seem so dreary to many. And we take an embodied stance on creativity in math, arguing that much of what can be called creative in this subject area arises from spatial mental imagery and the sensorimotor grounding of mathematical concepts. We discuss numerous experiments on mathematical metaphors such as MORE IS UP, MORE IS RIGHT and MORE IS BIGGER. Together, these studies support the existence of connections between our sense of number and our sense of space. Alongside these experimental studies, we discuss how the interplay between mathematics and the arts provides evidence for the spatial nature of creativity in mathematics.

2 Mathematics, art and space

Many people are unaware of the intricate connections mathematics has with the arts. Sometimes mathematics is reflected in art, and sometimes art leads to the discovery of new mathematics. In this section, we discuss examples of this interplay between these seemingly different fields, and we point out that a common theme that connects mathematics and the arts is their involvement in space and spatial creativity. After that, we discuss experimental evidence supporting a mental connection between math and space.

One example of the link between creativity and mathematics is seen in Renaissance art. During this epoch, artists became increasingly interested in representing the real world in spatially accurate perspectives. To do so, artists had to develop principles that would enable the projection of three-dimensional space onto the two-dimensional space of a canvas. This, in turn, led to the development of an entirely new branch of mathematics: projective geometry. Kline (1967: 230) says that "The works of the Renaissance artists are hung in art museums. They could, with as much justification, be hung in science museums." Thus, projective geometry is a branch of mathematics that can arguably be seen as stemming from the creative exploration.

Another example of the intricate connections between space, art and mathematics is "hyperbolic crocheting". Hyperbolic spaces are an essential part of Non-Euclidian geometry, but Bellos (2010: 383) rightly calls hyperbolic geometry "an utterly counter-intuitive type of geometry". In fact, it has been claimed that it is impossible to build a stable real-world model of hyperbolic spaces that could be used to demonstrate how these spaces look like (cf. discussion in Henderson & Taimiða, 2001). Because of this, it was hard to come up with a consistent mental image of how a hyperbolic space might look like, making the topic elusive and literally difficult to grasp. But then, the Latvian mathematician Daina Taimiða stumbled upon a way of constructing these spaces *while crocheting*. She discovered that hyperbolic spaces can be created following certain stitching patterns, an idea that was later formalized in Henderson and Taimiða (2001). This example demonstrates how something that was previously unimaginable and abstract has been made concrete and tangible through the creative exercise of crocheting.

A final example of the development of new mathematics through art is origami, which recently has underwent a resurgence of interest due to the development of computational origami (e.g., Demaine & O'Rourke, 2007), a field that has great prospects of producing efficient engineering and design structures that use minimal space when folded, but that can serve numerous useful functions when expanded. Computational origami builds on a set of axioms that were discovered by Humiaki Huzita, a Japanese-Italian mathematician and origami artist. Again, we can see that a form of art (origami) drove the development and discovery of new mathematics.

Across these different examples it is clear that arts and mathematics are linked through their common involvement with space. Other examples of this connection include the mathematical art of M.C. Escher, the mathematics of tessellation in Arab mosaics, fractal art, and the aesthetically pleasing visuospatial proofs of the Pythagorean theorem (see Nelsen, 1993).

It could be argued that art and mathematics cross-foster each other only when both already share the common denominator of space. However, this common denominator sometimes needs to be cognitively created afresh, which is a creative act. A case in point is the number *i* (the square root of negative one). For most of its history, *i* has been elusive and its status as a number was controversial. Lakoff and Núñez (2000, ch. VI:3), Mazur (2003) and Fauconnier and Turner (2001) argue that the concept of the number *i* becomes graspable once one mentally projects *i* and its multiples onto a number plane. Mazur (2003) argues that the creative act of coming up with the spatial image of *i* being located on a number plane played a pivotal role in the cultural acceptance of this concept. Moreover, the spatial view of this number unlocks new mathematics (such as polar coordinate geometry) which can then be applied to *i*.

To conclude this section, it seems that the history of mathematics, as well as the history of arts, support the idea that spatial mental imagery plays a role in leading to novel and ultimately creative insights in mathematics.

3 Embodied mathematics and experiments

Lakoff and Núñez (2002) use Conceptual Metaphor Theory (Lakoff & Johnson, 1980; Gibbs, 1994) to analyze mathematical ideas (see also Núñez, 2005; 2007). They propose that mathematics can be seen as a system of metaphorical mappings such as ARITHMETIC IS OBJECT COLLECTION or ARITHMETIC IS MOTION ALONG A PATH. Alongside this research, mathematicians have also discussed how some of the seemingly abstract axioms at the foundation of mathematics (e.g., negative times negative yields positive) are grounded in concrete physical principles (Kline, 1967: 371).

Alongside this more theoretical perspective, psychologists, cognitive scientists and neuroscientists have explored experimentally whether there are mental connections between space and mathematics. Dehaene, Bossini and Giraux (1993) demonstrated that Western Europeans seem to think of numbers as being aligned on a left-to-right going number line. These researchers found that when participants saw relatively larger numbers on a screen (say, 8 instead of 2), they were faster to press a button with their right hand than with their left hand. This is interpreted as showing that if the response is consistent with a horizontal mental number line, responses are sped up. This general finding has been replicated in more than 100 experiments (see review in Wood, Nuerk, Willmes & Fischer, 2008), and it can be taken to suggest a mapping that can be called MORE IS RIGHT.

Other studies support a vertical mapping, in line with the metaphor MORE IS UP (cf., Lakoff, 1987: 276-277). For example, Hartmann, Grabherr and Last (2011) found that participants generated "higher" numbers when their bodies were moved upwards, and "lower" numbers when their bodies were moved downwards. In a similar vein, Winter and Matlock (2013) found that when participants were asked to call out numbers while they moved their head rhythmically upwards and downwards, they generated larger numbers when looking upwards. Sell and Kaschak (2012) found that when participants read sentences that contained quantity information such as the words "more" or "less", they were quicker to respond with an upwards oriented response button to larger implied quantities, and with a downwards oriented response button to smaller implied quantities.

Another mapping that has gained experimental support is MORE IS BIGGER, or QUANTITY IS SIZE. For example, when participants were shown relatively "large" numbers (numerically speaking), they were quicker to respond if the number was presented in larger font (Henik & Tzelgov, 1982). Hurewitz, Gelman and Schnitzer (2006) found participants to systematically overestimate the quantity of dots if these covered more area. These results show that the mental concepts of size and numerical quantity are interacting with each other, in line with a metaphorical mapping of physical extent to numerical quantity.

While these studies provide evidence for conceptual mappings such as MORE IS RIGHT, MORE IS UP or MORE IS BIGGER, there is abundant evidence for mental connections between space and numbers in the field of neuroscience. Many neuroscientific studies on this topic have shown that the parietal cortex becomes activated when participants process numbers or quantities (Hubbard, Piazza, Pinel, & Dehaene, 2005; Pinel, Piazza, Le Bihan, & Dehaene, 2004). This brain area is not only implicated in the processing of number and quantity, it is also implicated in the processing of space, therefore, indicating that the mental connection between space and numbers is mirrored by a neurological connection as well.

Compared to the high-level interactions between the arts, mathematics and space discussed in section 3, the experimental results discussed in this section are located at a much lower level, focusing on simple things such as mental arithmetic or numerical representation. The connection between abstract mathematics and simple numerical representations still needs to be made. However, initial evidence already supports the idea that the two levels are connected, and that spatial thinking in abstract mathematics builds on spatial thinking about basic numbers and simple mathematical operations. For example, when mathematics student explain abstract proofs from calculus, they tend to make co-speech gestures (see, e.g., Marghetis & Núñez, 2013), many of which are in line with the basic numerical representations that are discussed in the experimental studies mentioned above.

4 MORE IS BIGGER in mental arithmetic

Both for abstract mathematical concepts and the mental representation of numbers, spatial thinking has been claimed to play a heavy role. But, currently there is not much experimental evidence to support that idea that the spatial concepts discussed above play a role in *doing* math. In this section, we discuss a survey-based study that explores the role of the conceptual metaphor MORE IS BIGGER in performing simple arithmetic problems. The study seeks evidence for the activation of size-based concepts while computing routine addition and subtraction. A total of 203 University of California, Merced, undergraduates took part in the study and received extra credit in an undergraduate social sciences course. All participants were native speakers of English. Each received a two-page study, of which the first page contained the following instructions:

> "On the next page, you will see a simple math equation above a box. Inside the box, you will see "=" preceded by the operator symbol "+" or "-". Your job is to fill in the blanks by drawing circles to match the numerical values in the math equation above the box. For example, for the number "6", you would draw 6 circles. Please draw your circles as quickly as possible. You can turn the page now."

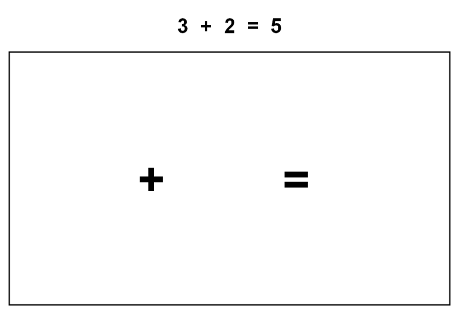

Fig. 1: Example stimulus (addition condition) from the drawing experiment.

There were two conditions. Each participant was either in an addition condition (they saw either 3+2=5 or 2+2=4), or in a subtraction condition (they saw either 7-2=5 or 6-2=4). We predicted that addition should tacitly trigger the numerical concept "more" because it represents an increase in quantity, and subtraction should trigger the concept "less" because it represents a decrease in quantity. If these concepts are indeed mentally related to conceptions of size, differences in the *physical* size of the circles should arise. In other words, it should be the case that participants draw slightly larger outcome circles for addition than for subtraction.

We measured the rectangular area covered by circle groups by multiplying the largest horizontal extent of the group of circles with the largest vertical extent. Participants drew circle groups in clusters (127 participants) or horizontally lines (76). However, the interpretation of the results that we report below did not differ between cluster-drawers and line-drawers, thus we only report analyses that combined these two drawing strategies[1].

The results are displayed in Fig. 2. Participants in our study drew circle groups about 3.04 square centimeter larger in the addition condition (M=5.27) than participants in the subtraction condition (M=8.31) ($t(191.784)=4.25$, $p<0.001$). This is in line with the idea that people naturally think about an increase in quantity, and concomitantly, about an increase in physical size when they are doing addition. So, even though the circle size is an irrelevant dimension with respect to solving this task (critically, we never asked people to pay attention to the size of the circles), we observed differences in area that are consistent with the idea that MORE IS BIGGER plays a role in mental arithmetic. These findings are also consistent with the idea that people use metaphors such as ARITHMETIC IS OBJECT COLLECTION (Lakoff & Núñez, 2000) to reason about operations such as addition and subtraction.

[1] Only 12 participants drew circles arranged as vertical lines. Because these were too few to be analyzed separately, we excluded these 12 data points (6% of the overall data) from the analysis. All other items (215) were considered for the analysis. The four distances were measured by a research assistant. To assure reliability of measurements, the first author recoded 10% of the items. The two measurements correlated well with each other (r=0.89) and the average deviation between coders was low (~1.11 mm).

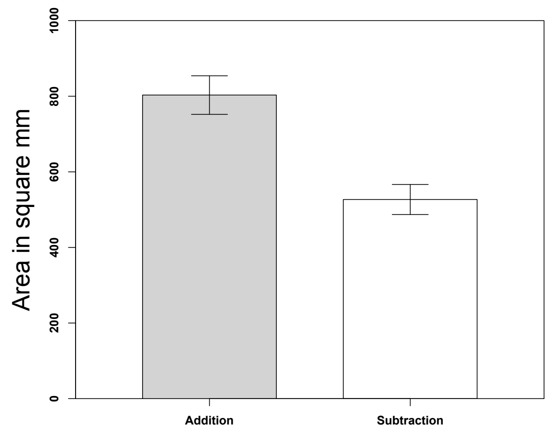

Fig. 2: Overall circle area in addition and subtraction condition. Error bars represent standard errors.

It is worth noting that we only discovered an effect of size, and not an overall dislocation of the group of circles. For example, if participants were thinking about arithmetic in terms of motion along a path in this experiment, we would have expected the group of circles to be dislocated rightwards with addition (consistent with the metaphor MORE IS RIGHT) and leftwards with subtraction. However, this is not what we found. Apparently participants were more likely to respond in a way that is consistent with MORE IS BIGGER in this task. There are several reasons why this could be the case. First, MORE IS RIGHT is not expressed in English, but MORE IS BIGGER is. When we talk about numbers, we either use the language of size ("this is a huge sum") or the language of verticality ("this is a high number"), but we do not say "this number is more towards the right" to say that a number is larger, except for maybe some really special circumstances (e.g., when talking about visually represented sequences). However, if language entrenchment were the only factor, we would also expect to find the circles to be physically higher in the addition condition because of the prevalence of talking about numbers in terms of verticality. This is not what we found.

Another reason why participants were more likely to draw larger outcome circles after addition rather than higher outcome circles might have to do with the fact that circles can be more readily interpreted as object collections, such as a pile of gumballs or pile of popcorn. When the quantity of such collections is varied, the overall size of the collection is expected to change, but not the horizontal or vertical dislocation of the collection. Thus, the task that we employed might be more readily interpreted in lines with

ARITHMETIC IS OBJECT COLLECTION and the consistent metaphor MORE IS BIGGER, than with ARITHMETIC IS MOTION ALONG A PATH. Which metaphors structure our thinking about arithmetic probably relies to some extent on the nature of the stimuli used and the nature of the task.

We have to admit, however, that there is an important alternative explanation for the data presented above. It could potentially be the case that we observed smaller circles after subtraction simply because with our subtraction items, the first number in the sequence was necessarily larger (**7**-2=5, **6**-2=4) than with addition (**3**+2=5, **2**+2=4). Maybe participants were drawing smaller outcome circles because these seemed smaller compared to 6 or 7. To disentangle our current hypothesis from this alternative explanation, future work needs to test zero addition and subtraction such as 5+0=5 and 5-5=0.

5 Implications and Conclusions

We discovered that when people drew circles to depict numbers in arithmetic problems, they drew larger circle groups for addition than for subtraction. Overall, these results lend support to the idea that mathematics imports conceptual structure from physical experiences, in line with such metaphorical mappings as MORE IS BIGGER. From this perspective, mathematics is in fact not that abstract, but instead deeply connected to concrete notions such as physical size.

Language reflects this as well: We use the language of physical size to talk about numerical quantities, such as when we say, "This is a huge sum" or "This is a large number". And it seems to be the case that a number of quantity terms can be historically traced back to words that describe size or physical extent, for example, the word "more" goes back to the Proto-Indo-European root **me-* "big", and "quantity" goes back to the Latin word *quantitatem* "relative greatness or extent" (Harper, 2012). So, our experiment (section 4) and other experiments on mathematical cognition (section 3) mesh nicely with these observations from synchronic and diachronic linguistics. There seems to be a lot of converging evidence that suggests that math is in fact not that abstract.

What are the implications of this embodied perspective of mathematics? In this section, we will discuss how this approach can be helpful in teaching mathematics, and how it can be used to get more people to become interested in this subject. Many people see mathematics as dreary or daunting. Some people actively try to steer away from this subject, and some of these people can even be classified as "innumerates", as mathematical illiterates (Paulos, 1988). Some people even have innate or neurological mathematical disorders, such as dyscalculia. Yet, in our everyday lives, we constantly need to add and subtract quantities, for example, when checking whether we have enough money to buy doughnuts and coffee at the diner, or when planning dinner parties. Moreover, much of our everyday work life requires us to perform simple calculations with numbers and quantities. How might embodied mathematics be helpful here?

If people really do think about numbers in terms of space, then we should try to foster these thought patterns. Bryant and Squire (2001: 175) mention that many developmental psychologists viewed the connection between space and numbers "in a negative way". They say that space for these psychologists "is part of the problem in children's mathematics, not part of the solution" (cf. discussion in Walsh, 2003). An embodied perspective would argue that the way we teach mathematics should be in line with our "natural mappings". Learning and teaching should be consistent with our natural tendencies to construe the world, and it should follow the natural way in which the human mind develops (Egan, 1997). Thus, if we acquire spatial metaphors of numbers such as MORE IS UP and MORE IS BIGGER through embodied interaction with the world early on in our lives, we should embed these metaphors in teaching rather than trying to avoid them.

Some strands of mathematical education are already in line with this idea. For example, many mathematical textbooks for kindergarteners and first graders are colorful and full of concrete physical examples, and a lot of them reinforce such important concepts as the number line (consider the Number Line Frog that appears in many American textbooks). However, it seems that once children proceed to middle school and high school mathematics, a lot of this concrete approach is given up, even though, as Lakoff and Núñez (2000) argue, much of "higher" mathematics seems to also rely on spatial mental imagery and other embodied concepts. It seems that this abstract way of teaching mathematics might unduly limit the creative exploration of this subject matter.

Middle school and high school mathematics teaching is typically fairly disembodied. However, in recent years, there is a growing interesting in actively pushing a more embodied perspective in mathematics education (see also, Núñez, Edwards, & Matos, 1999; Núñez, 2007). For example, the Mathematical Imagery Trainer (Howison, Trninic, Reinholz, & Abrahamson, 2011; Euson & Abrahamson, 2005) uses movements in line with MORE IS UP to teach fractions and proportional progression (2:3 – 4:6 – 6:9). And, Cress, Fischer, Moeller, Sauter and Nuerk (2010) used a digital dance mat to teach relative numerical magnitudes. And, more and more teaching tools that promote numeracy in patients with dyscalculia start emphasizing space to increasing extents (Wilson, Dehaene, Pinel, Revkin, Cohen & Cohen, 2006).

We offer some ideas here. First, the involvement of mathematics in art and the involvement of art in mathematics (discussed in section 2) could be used as another avenue in making mathematics more interesting and more graspable. Adults and children who cannot or do not want to realize the beauty of math themselves might be motivated if confronted with some of the many examples that highlight the more creative aspects of mathematics. Zagier (2012: 16) points out that the "mathematics that everyone learns in school is almost always just a collection of recipes for everyday use or, at best, in science." Some of the most creative, interesting and aesthetically pleasing aspects of mathematics are never discussed in school mathematics. Even though some examples discussed in section 2 might not be immediately relevant for specific applications (e.g., leading towards calculus or linear algebra, topics important in engineering), they might be useful in keeping students interested in mathematics and making it seem like a topic that is less removed and abstract than is generally believed.

Second, given that we already have evidence for MORE IS BIGGER, MORE IS UP and MORE IS RIGHT, metaphor research needs to address such questions as: What are the specific advantages conferred by these different metaphors? Does structuring number space in one way as opposed to another help in certain situations? Do different people have different propensities to think about numbers in one way or another? And, under which conditions does which metaphor become salient? We also need to investigate further how these relatively low-level metaphors relate to more abstract and high-level mathematics such as calculus and linear algebra. Addressing these questions will be crucial for developing more advanced and more targeted embodied teaching tools.

Zagier (2012: 12) contrasts mathematics from other fields by pointing out that while other sciences "are clearly characterized according to the objects they study: heavenly bodies, living things, human relationships", mathematics studies many different objects, some of which seem – at first sight – fairly abstract. For example, entities such as "sets", "equations" or "imaginary numbers" appear to have no straightforward connection to the real world. However, when there is a lack of concrete physical objects, education can create these objects, sometimes in a physical form, such as in hyperbolic crocheting, and sometimes in a mental form, such as in the "imaginary number line" of *i*. In this paper, we have argued that creating such mental objects can aid mathematical thinking, and more generally, we have argued that an embodied approach will prove useful for teaching mathematics.

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